On equi-Weyl almost periodic selections of multivalued maps

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Abstract

We prove that equi-Weyl almost periodic multivalued maps $\mathbf{R} \ni t \to F(t) \in \mathrm{cl}\,\mathcal{U}$ have equi-Weyl almost periodic selections, where $\mathrm{cl}\,\mathcal{U}$ is the collection of non-empty closed sets of a complete metric space $\mathcal{U}$.

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Introduction

Let $(\mathcal{U}, \rho)$ be a complete metric space and let $(\mathrm{cl}_b\,\mathcal{U}, \mathrm{dist})$ be the metric space of non-empty closed bounded sets $A \subset \mathcal{U}$ with the Hausdorff metric

$$\mathrm{dist}(A,B) = \mathrm{dist}_\rho(A,B) = \max\left\{\sup_{x\in A}\rho(x,B), \sup_{x\in B}\rho(x,A)\right\},\ \ A, B \in \mathrm{cl}_b\,\mathcal{U}\,,$$

where $\rho(x,F) = \inf_{y\in F}\rho(x,y)$ is a distance from a point $x \in \mathcal{U}$ to a non-empty set $F \subset \mathcal{U}$. The metric space $(\mathrm{cl}_b\,\mathcal{U}, \mathrm{dist})$ is also complete. Let $\mathrm{cl}\,\mathcal{U}$ be the collection of non-empty closed sets $A \subset \mathcal{U}$. The closure of a set $A \subset \mathcal{U}$ will be denoted by $\overline{A}$.

In this paper we prove the existence of equi-Weyl almost periodic (a.p.) selections of equi-Weyl a.p. multivalued maps $\mathbf{R} \ni t \to F(t) \in \mathrm{cl}\,\mathcal{U}$. These results are used in the study of a.p. solutions of differential inclusions [1, 2].

It is known that the Bohr a.p. multivalued maps $\mathbf{R} \ni t \to F(t) \in \mathrm{cl}_b\,\mathcal{U}$ need not have Bohr a.p. selections. In particular, in a finite-dimensional Euclidean space it is not always possible to supplement a Bohr a.p. orthogonal frame to form a Bohr a.p. orthogonal basis [3]. It is also not hard to construct an example of continuous function $\mathbf{R} \ni t \to F(t) \in \mathbf{R}^2\backslash\{0\}$ which is not Bohr a.p. such that $\mathbf{R} \ni t \to \{f(t), -f(t)\} \in \mathrm{cl}_b\,\mathbf{R}^2$ is a two-valued Bohr a.p. map [4].

In [5], in the case of a separable Banach space $\mathcal{H}$ it has been proved that for any Stepanov a.p. (of degree 1) multivalued map $F \in S_1(\mathbf{R}, \mathrm{cl}_b\mathcal{H})$ there exists

a countable set of Stepanov a.p. (of degree 1) selections $f_j \in S_1(\mathbf{R},\mathcal{H})$, $j \in \mathbf{N}$, such that $\operatorname{Mod} f_j \subset \operatorname{Mod} F$ and $F(t) = \overline{\bigcup_j f_j(t)}$ for almost every (a.e.) $t \in \mathbf{R}$. The Stepanov a.p. selections of multivalued maps $\mathbf{R} \ni t \to F(t) \in \mathrm{cl}\,\mathcal{U}$ have been studied in [4], [6] – [11]. In [11], in particular, for a compact metric space $\mathcal{U}$ it has been proved that a multivalued map $\mathbf{R} \ni t \to F(t) \in \mathrm{cl}_b\mathcal{U}$ belongs to the space $S_1(\mathbf{R},\mathrm{cl}_b\mathcal{U})$ if and only if there exists a countable set of selections $f_j \in S_1(\mathbf{R},\mathcal{U})$, $j \in \mathbf{N}$, for which $F(t) = \overline{\bigcup_j f_j(t)}$ almost everywhere (a.e) and $\{f_j(.) : j \in \mathbf{N}\}$ is a precompact set in $L_\infty(\mathbf{R},\mathcal{U})$ (furthermore, the selections $f_j \in S_1(\mathbf{R},\mathcal{U})$ of the multivalued map $F(.) \in S_1(\mathbf{R},\mathrm{cl}_b\mathcal{U})$ can be chosen such that $\operatorname{Mod} f_j \subset \operatorname{Mod} F$).

In the proofs suggested in this paper we use the modifications for equi-Weyl case of some results from [4] and [8].

In Section 1 we give definitions and formulate our main results. Some assertions, which will be used throughout what follows, about a.p. functions are contained in Section 2. The simple and well known assertions (see e.g. [12]) are given without proofs. In the subsequent Sections we prove the Theorems from Section 1.

1 Definitions and main results

Let meas be Lebesgue measure on $\mathbf{R}$ and let $B_r(x) = \{y \in \mathcal{U} : \rho(x,y) \le r\}$, $x \in \mathcal{U}$, $r \ge 0$. A function $f : \mathbf{R} \to \mathcal{U}$ is said to be *elementary* if there exist points $x_j \in \mathcal{U}$ and disjoint measurable (in the Lebesgue sense) sets $T_j \subset \mathbf{R}$, $j \in \mathbf{N}$, such that $\operatorname{meas} \mathbf{R}\backslash \bigcup_j T_j = 0$ and $f(t) = x_j$ for all $t \in T_j$. We denote this function by $f(.) = \sum_j x_j \chi_{T_j}(.)$ (where $\chi_T(.)$ is the characteristic function of a set $T \subset \mathbf{R}$). For arbitrary functions $f_j : \mathbf{R} \to \mathcal{U}$, $j \in \mathbf{N}$, we define the function $\sum_j f_j(.)\chi_{T_j}(.) : \mathbf{R} \to \mathcal{U}$ coinciding with $f_j(.)$ on the set T_j for $j \in \mathbf{N}$ (in the case of the metric space $\mathcal{U}$ the notations used are formally incorrect, but no linear operations will be carried out on the functions under consideration). A function $f : \mathbf{R} \to \mathcal{U}$ is *measurable* if for any $\epsilon > 0$ there exists an elementary function $f_\epsilon : \mathbf{R} \to \mathcal{U}$ such that

$$\operatorname*{ess\,sup}_{t\in\mathbf{R}} \rho(f(t), f_\epsilon(t)) < \epsilon\,.$$

The class of measurable functions $f : \mathbf{R} \to \mathcal{U}$ will be denoted by $M(\mathbf{R},\mathcal{U})$ (functions coinciding for a.e. $t \in \mathbf{R}$ will be identified). Let a point $x_0 \in \mathcal{U}$ be fixed. We use the notation

$$M_p(\mathbf{R},\mathcal{U}) \doteq \{f \in M(\mathbf{R},\mathcal{U}) : \sup_{\xi\in\mathbf{R}} \int\limits_{\xi}^{\xi+1} \rho^p(f(t),x_0)\,dt < +\infty\}\,,\ p \ge 1\,,$$

and for all $l > 0$ we define the metrics on $M_p(\mathbf{R},\mathcal{U})$:

$$D^{(\rho)}_{p,l}(f,g) = \left(\sup_{\xi\in\mathbf{R}} \frac{1}{l} \int\limits_{\xi}^{\xi+l} \rho^p(f(t),g(t))\,dt \right)^{1/p}, \ f,g \in M_p(\mathbf{R},\mathcal{U})\,.$$

If $l_1 \geq l$, then

$$\left(\frac{l}{l_1}\right)^{1/p} D^{(\rho)}_{p,l}(f,g) \leq D^{(\rho)}_{p,l_1}(f,g) \leq \left(1+\frac{l}{l_1}\right)^{1/p} D^{(\rho)}_{p,l}(f,g)\,,$$

therefore, the metrics $D^{(\rho)}_{p,l}$, $l > 0$, are equivalent and there exists the limit

$$D^{(\rho),W}_{p}(f,g) = \lim_{l\to+\infty} D^{(\rho)}_{p,l}(f,g) = \inf_{l>0} D^{(\rho)}_{p,l}(f,g)\,, \ f,g \in M_p(\mathbf{R},\mathcal{U})\,.$$

For a Banach space $\mathcal{U} = (\mathcal{H},\|.\|)$ ($\rho(x,y) = \|x-y\|$, $x,y \in \mathcal{H}$) we denote by

$$\|f\|_{p,l} = \left(\sup_{\xi\in\mathbf{R}} \frac{1}{l} \int\limits_{\xi}^{\xi+l} \|f(t)\|^p dt \right)^{1/p}, \ l > 0\,,$$

and by

$$\|f\|_p = \lim_{l\to+\infty} \|f\|_{p,l}$$

the norms and the seminorm on the linear space $M_p(\mathbf{R},\mathcal{H}) \ni f$, $p \geq 1$. In what follows, it is convenient to assume the Banach space $(\mathcal{H},\|.\|)$ to be complex. If the Banach space $\mathcal{H}$ is real, then we can consider the complexification $\mathcal{H}+i\mathcal{H}$ identifying the space $\mathcal{H}$ with the real subspace (the norm $\|.\|_{\mathcal{H}+i\mathcal{H}}$ on the real subspace coincides with the norm $\|.\|$).

A set $T \subset \mathbf{R}$ is called *relatively dense* if there exists a number $a > 0$ such that $[\xi,\xi+a]\cap T \neq \emptyset$ for all $\xi \in \mathbf{R}$. A continuous function $f \in C(\mathbf{R},\mathcal{U})$ belongs to the space $CAP\,(\mathbf{R},\mathcal{U})$ of *Bohr a.p.* functions if for any $\epsilon > 0$ there exists a relatively dense set $T \subset \mathbf{R}$ such that the inequality

$$\sup_{t\in\mathbf{R}} \rho(f(t),f(t+\tau)) < \epsilon$$

holds for all $\tau \in T$. A number $\tau \in \mathbf{R}$ is called an $(\epsilon, D^{(\rho)}_{p,l})$-*almost period* of function $f \in M_p(\mathbf{R},\mathcal{U})$, $\epsilon > 0$, if $D^{(\rho)}_{p,l}(f(.),f(.+\tau)) < \epsilon$. A function $f \in M_p(\mathbf{R},\mathcal{U})$, $p \geq 1$, belongs to the space $S_p(\mathbf{R},\mathcal{U})$ of *Stepanov a.p.* functions *of degree* p if for any $\epsilon > 0$ the set of $(\epsilon, D^{(\rho)}_{p,1})$-almost periods of f is relatively dense. A function $f \in M_p(\mathbf{R},\mathcal{U})$, $p \geq 1$, belongs to the space $W_p(\mathbf{R},\mathcal{U})$ of *equi-Weyl a.p.* functions *of degree* p if for any $\epsilon > 0$ there exists a number

$l = l(\epsilon, f) > 0$ such that the set of $(\epsilon, D^{(\rho)}_{p,l})$-almost periods of f is relatively dense. We have the inclusions $CAP\,(\mathbf{R},\mathcal{U}) \subset S_p(\mathbf{R},\mathcal{U}) \subset W_p(\mathbf{R},\mathcal{U})$.

The a.p. functions $f \in W_p(\mathbf{R},\mathcal{U})$ are called equi-Weyl to distinguish them from Weyl a.p. functions (a function $f \in M_p(\mathbf{R},\mathcal{U})$ is said to be Weyl a.p. function if for any $\epsilon > 0$ there is a relatively dense set $T \subset \mathbf{R}$ such that $D^{(\rho),W}_p(f(.), f(.+\tau)) < \epsilon$ for all $\tau \in T$). In [12], the functions $f \in W_p(\mathbf{R},\mathcal{U})$ are called Weyl almost periodic.

A sequence $\tau_j \in \mathbf{R}$, $j \in \mathbf{N}$, is said to be f*-returning* for a function $f \in W_p(\mathbf{R},\mathcal{U})$ if for any $\epsilon > 0$ there exist numbers $l = l(\epsilon, f) > 0$ and $j_0 \in \mathbf{N}$ such that all numbers $\tau_j : j \geq j_0$ are $(\epsilon, D^{(\rho)}_{p,l})$-almost periods of function f.

If $f \in S_p(\mathbf{R},\mathcal{U}) \subset W_p(\mathbf{R},\mathcal{U})$, then a sequence $\tau_j \in \mathbf{R}$, $j \in \mathbf{N}$, is f-returning if and only if $D^{(\rho)}_{p,1}(f(.), f(.+\tau_j)) \to 0$ as $j \to +\infty$. If $f \in CAP\,(\mathbf{R},\mathcal{U}) \subset W_p(\mathbf{R},\mathcal{U})$, then a sequence $\tau_j \in \mathbf{R}$, $j \in \mathbf{N}$, is f-returning if and only if

$$\sup_{t\in\mathbf{R}} \rho(f(t), f(t+\tau_j)) \to 0$$

as $j \to +\infty$.

For a function $f \in W_p(\mathbf{R},\mathcal{U})$ we denote by $\mathrm{Mod}\, f$ the set of numbers $\lambda \in \mathbf{R}$ for which $e^{i\lambda\tau_j} \to 1$ $(i^2 = -1)$ as $j \to +\infty$ for all f-returning sequences τ_j. The set $\mathrm{Mod}\, f$ is a module (additive group) in $\mathbf{R}$. If $D^{(\rho),W}_p(f(.), f_0(.)) \neq 0$ for all constant functions $f_0(t) \equiv f_0 \in \mathcal{U}$, $t \in \mathbf{R}$, then $\mathrm{Mod}\, f$ is a countable module ($\mathrm{Mod}\, f = \{0\}$ otherwise).

On the space $\mathcal{U}$ we also consider the metric $\rho'(x,y) = \min\{1, \rho(x,y)\}$, $x, y \in \mathcal{U}$. The metric space $(\mathcal{U}, \rho')$ is complete as well as $(\mathcal{U}, \rho)$. For all $f, g \in M(\mathbf{R},\mathcal{U}) = M_1(\mathbf{R}, (\mathcal{U}, \rho'))$ we use the notations

$$D^{(\rho)}_l(f,g) = D^{(\rho')}_{1,l}(f,g) = \sup_{\xi\in\mathbf{R}} \frac{1}{l} \int\limits_{\xi}^{\xi+l} \rho'(f(t), g(t))\, dt\,,\ l > 0\,,$$

$$D^{(\rho),W}(f,g) = \lim_{l\to+\infty} D^{(\rho)}_l(f,g)\,.$$

Let $S(\mathbf{R},\mathcal{U}) \doteq S_1(\mathbf{R}, (\mathcal{U}, \rho'))$, $W(\mathbf{R},\mathcal{U}) \doteq W_1(\mathbf{R}, (\mathcal{U}, \rho'))$. We have $S(\mathbf{R},\mathcal{U}) \subset W(\mathbf{R},\mathcal{U})$, $W_p(\mathbf{R},\mathcal{U}) \subset W(\mathbf{R},\mathcal{U})$, $p \geq 1$. A sequence $\tau_j \in \mathbf{R}$, $j \in \mathbf{N}$, is f-returning for a function $f \in W_p(\mathbf{R},\mathcal{U})$ if and only if it is f-returning for a function f considered as the element of the space $W(\mathbf{R},\mathcal{U})$ (the set of f-returning sequences is determined only by the a.p. function itself and does not depend on the spaces under consideration of a.p. functions that include the function f).

We shall denote by $W(\mathbf{R})$ the collection of measurable (in the Lebesgue sense) sets $T \subset \mathbf{R}$ such that $\chi_T \in W_1(\mathbf{R}, \mathbf{R})$. For a set $T \in W(\mathbf{R})$ let $\mathrm{Mod}\, T = \mathrm{Mod}\, \chi_T$.

For an arbitrary module (additive group) $\Lambda \subset \mathbf{R}$ let $\mathfrak{M}^{(W)}(\Lambda)$ be the set of sequences $\{T_j\}_{j\in\mathbf{N}}$ of disjoint sets $T_j \in W(\mathbf{R})$ such that $\operatorname{Mod} T_j \subset \Lambda$, $\operatorname{meas} \mathbf{R}\backslash\bigcup_j T_j = 0$ and $\|\chi_{\mathbf{R}\backslash\bigcup_{j\le N} T_j}(.)\|_1 \to 0$ as $N \to +\infty$. We shall assume that the set $\mathfrak{M}^{(W)}(\Lambda)$ includes the corresponding finite sequences $\{T_j\}_{j=1,\dots,N}$ as well, which can always be supplemented by empty sets to form denumerable sequences. The sets T_j of sequences $\{T_j\} \in \mathfrak{M}^{(W)}(\Lambda)$ will also be enumerated by means of several indices.

If $\Lambda_j \subset \mathbf{R}$ are arbitrary modules, then by $\sum_j \Lambda_j$ (or by $\Lambda_1 + \dots + \Lambda_n$ for finitely many modules Λ_j , $j = 1, \dots, n$) we denote the sum of modules, that is, the smallest module (additive group) in $\mathbf{R}$ containing all the sets Λ_j .

Theorem 1.1. *Suppose that $\{T_j\} \in \mathfrak{M}^{(W)}(\mathbf{R})$ and $f_j \in W(\mathbf{R},\mathcal{U})$ for $j \in \mathbf{N}$. Then*

$$\sum_j f_j(.)\chi_{T_j}(.) \in W(\mathbf{R},\mathcal{U})$$

and

$$\operatorname{Mod} \sum_j f_j(.)\chi_{T_j}(.) \subset \sum_j \operatorname{Mod} f_j + \sum_j \operatorname{Mod} T_j \,. \tag{1}$$

Remark 1. Under the assumptions of Theorem 1.1 for indices $j \in \mathbf{N}$ such that $\|\chi_{T_j}\|_1 = 0$ (in this case $\operatorname{Mod} T_j = \{0\}$) we can choose arbitrary functions $f_j \in M(\mathbf{R},\mathcal{U})$ and delete these indices in the summation on the right-hand side of inclusion (1).

The following Theorem is analogous to the Theorem on uniform approximation of Stepanov a.p. functions [4, 9, 10] and plays a key role in this paper.

Theorem 1.2. *Let $f \in W(\mathbf{R},\mathcal{U})$. Then for any $\epsilon > 0$ there exist a sequence $\{T_j\} \in \mathfrak{M}^{(W)}(\operatorname{Mod} f)$ and points $x_j \in \mathcal{U}$, $j \in \mathbf{N}$, such that $\rho(f(t), x_j) < \epsilon$ for all $t \in T_j$, $j \in \mathbf{N}$.*

Theorem 1.2 is proved in Section 3.

Let $\operatorname{dist}_{\rho'}$ be the Hausdorff metric on $\operatorname{cl}\mathcal{U} = \operatorname{cl}_b(\mathcal{U}, \rho')$ corresponding to the metric ρ'. The metric space $(\operatorname{cl}\mathcal{U}, \operatorname{dist}_{\rho'})$ is complete. Since $\operatorname{dist}'(A,B) \doteq \min\{1, \operatorname{dist}(A,B)\} = \operatorname{dist}_{\rho'}(A,B)$ for all $A, B \in \operatorname{cl}_b\mathcal{U}$, it follows that the embedding $(\operatorname{cl}_b\mathcal{U}, \operatorname{dist}') \subset (\operatorname{cl}\mathcal{U}, \operatorname{dist}_{\rho'})$ is isometric. We define the spaces $W(\mathbf{R}, \operatorname{cl}_b\mathcal{U})$ and $W_p(\mathbf{R}, \operatorname{cl}_b\mathcal{U})$, $p \ge 1$, of *equi-Weyl a.p. multivalued maps* $\mathbf{R} \ni t \to F(t) \in \operatorname{cl}_b\mathcal{U}$ as the spaces of equi-Weyl a.p. functions taking values in the metric space $(\operatorname{cl}_b\mathcal{U}, \operatorname{dist})$. Let $W(\mathbf{R}, \operatorname{cl}\mathcal{U}) \doteq W_1(\mathbf{R}, (\operatorname{cl}\mathcal{U}, \operatorname{dist}_{\rho'}))$. The following inclusions $W_p(\mathbf{R}, \operatorname{cl}_b\mathcal{U}) \subset W_1(\mathbf{R}, \operatorname{cl}_b\mathcal{U}) \subset W(\mathbf{R}, \operatorname{cl}_b\mathcal{U}) \subset W(\mathbf{R}, \operatorname{cl}\mathcal{U})$ hold.

Let us denote by $\mathcal{N}$ the set of non-decreasing functions $[0, +\infty) \ni t \to \eta(t) \in \mathbf{R}$ such that $\eta(0) = 0$ and $\eta(t) > 0$ for all $t > 0$.

Theorem 1.3. *Let $(\mathcal{U},\rho)$ be a complete metric space, let $F \in W(\mathbf{R},\mathrm{cl}\,\mathcal{U})$ and let $g \in W(\mathbf{R},\mathcal{U})$. Then for any function $\eta \in \mathcal{N}$ there exists a function $f \in W(\mathbf{R},\mathcal{U})$ such that $\mathrm{Mod}\, f \subset \mathrm{Mod}\, F + \mathrm{Mod}\, g$, $f(t) \in F(t)$ a.e. and $\rho(f(t),g(t)) \le \rho(g(t),F(t)) + \eta(\rho(g(t),F(t)))$ a.e. Moreover, if $F \in W_p(\mathbf{R},\mathrm{cl}_b\mathcal{U}) \subset W(\mathbf{R},\mathrm{cl}\,\mathcal{U})$, $p \ge 1$, then $f \in W_p(\mathbf{R},\mathcal{U})$.*

Theorem 1.3 is the main result of the paper on equi-Weyl a.p. selections of multivalued maps and is proved in Section 5.

2 Some properties of equi-Weyl a.p. functions

For functions $f,g \in M_p(\mathbf{R},\mathcal{U})$, $p \ge 1$, we set

$$J_p(f,g) = \lim_{\delta\to+0}\ \lim_{l_0\to+\infty}\ \sup_{l\ge l_0}\ \sup_{\xi\in\mathbf{R}} \left(\frac{1}{l} \sup_{T\subset[\xi,\xi+l]\,:\,\mathrm{meas}\,T\le\delta l} \int\limits_T \rho^p(f(t),g(t))\,dt\right)^{1/p};$$

$$J_p(f,g) \le \lim_{l_0\to+\infty}\ \sup_{l\ge l_0} D^{(\rho)}_{p,\,l}(f,g) = D^{(\rho),W}_p(f,g)\,.$$

We use the notation

$$M^{\sharp}_p(\mathbf{R},\mathcal{U}) = \{f \in M_p(\mathbf{R},\mathcal{U}) : J_p(f(.),x_0(.)) = 0\}\,,$$

where $x_0(t) \equiv x_0$, $t \in \mathbf{R}$. The set $M^{\sharp}_p(\mathbf{R},\mathcal{U})$ does not depend on the choice of the point $x_0 \in \mathcal{U}$.

The following simple Lemmas are used in the proof of Theorem 2.1.

Lemma 2.1. *A function $f \in M(\mathbf{R},\mathcal{U})$ belongs to the space $W(\mathbf{R},\mathcal{U})$ of equi-Weyl a.p. functions if and only if for any $\epsilon,\delta > 0$ there exists a number $l > 0$ such that the inequality*

$$\sup_{\xi\in\mathbf{R}} \mathrm{meas}\,\{t \in [\xi,\xi+l] : \rho(f(t),f(t+\tau)) \ge \delta\} < \epsilon l\,.$$

holds for relatively dense set of numbers $\tau \in \mathbf{R}$. Furthermore, the sequence $\tau_j \in \mathbf{R}$, $j \in \mathbf{N}$, is f-returning for a function $f \in W(\mathbf{R},\mathcal{U})$ if and only if for any $\epsilon,\delta > 0$ there exist numbers $l > 0$ and $j_0 \in \mathbf{N}$ such that for all $j \in \mathbf{N} : j \ge j_0$

$$\sup_{\xi\in\mathbf{R}} \mathrm{meas}\,\{t \in [\xi,\xi+l] : \rho(f(t),f(t+\tau_j)) \ge \delta\} < \epsilon l\,.$$

For a function $f \in M(\mathbf{R},\mathcal{U})$ and a number $R \ge 0$ we define the function

$$\mathbf{R} \ni t \to f_R(x_0;t) = \begin{cases} f(t) & \text{if } \rho(f(t),x_0) \le R\,, \\ x_0 & \text{if } \rho(f(t),x_0) > R\,. \end{cases}$$

Lemma 2.2. *Let $f \in W_p(\mathbf{R},\mathcal{U})$, $p \geq 1$, $x_0 \in \mathcal{U}$. Then $D_p^{(\rho),W}(f(.), f_R(x_0;.)) \to 0$ as $R \to +\infty$.*

Theorem 2.1. *For all $p \geq 1$*

$$W_p(\mathbf{R},\mathcal{U}) = W(\mathbf{R},\mathcal{U}) \bigcap M_p^\sharp(\mathbf{R},\mathcal{U}) .$$

Proof. Let $f \in W_p(\mathbf{R},\mathcal{U})$. By Lemma 2.2 (in view of boundedness of functions $f_R(x_0;.)$),

$$J_p(f(.), x_0(.)) \leq J_p(f(.), f_R(x_0;.)) + J_p(f_R(x_0;.), x_0)) =$$
$$= J_p(f(.), f_R(x_0;.)) \leq D_p^{(\rho),W}(f(.), f_R(x_0;.)) \to 0$$

as $R \to +\infty$. Hence $W_p(\mathbf{R},\mathcal{U}) \subset M_p^\sharp(\mathbf{R},\mathcal{U})$, and therefore, $W_p(\mathbf{R},\mathcal{U}) \subset W(\mathbf{R},\mathcal{U}) \bigcap M_p^\sharp(\mathbf{R},\mathcal{U})$. To prove the reverse inclusion it is necessary to use Lemma 2.1. □

Let $(\mathcal{H}, \|.\|)$ be a complex Banach space. For any function $f \in W_p(\mathbf{R}, \mathcal{H})$ and any $\lambda \in \mathbf{R}$ there exists a limit

$$\lim_{a\to+\infty} \frac{1}{2a} \int_{-a}^{a} e^{-i\lambda t} f(t)\, dt = M\,\{e^{-i\lambda t} f\}$$

(the integral is defined in the sense of Bochner). We denote by $\Lambda\{f\}$ the set of Fourier exponents of a function $f \in W_p(\mathbf{R}, \mathcal{H}) : \Lambda\{f\} = \{\lambda \in \mathbf{R} : M\,\{e^{-i\lambda t} f\} \neq 0\}$. Let $\mathrm{Mod}\,(\Lambda\{f\})$ be *the module of the Fourier exponents* of a function f (that is, the smallest additive group in $\mathbf{R}$ containing the set $\Lambda\{f\}$).

Lemma 2.3. *Let $f \in W_p(\mathbf{R},\mathcal{U})$. Then for any sequence $\tau_j \in \mathbf{R}$, $j \in \mathbf{N}$, the following three conditions are equivalent:*

(1) $\{\tau_j\}$ is f-returning sequence;

(2) $D_p^{(\rho),W}(f(.), f(. + \tau_j)) \to 0$ as $j \to +\infty$,

(3) $e^{i\lambda\tau_j} \to 1$ as $j \to +\infty$ for all $\lambda \in \mathrm{Mod}\, f$.

If $\mathcal{U} = (\mathcal{H}, \|.\|)$, then $\mathrm{Mod}\, f = \mathrm{Mod}\,(\Lambda\{f\})$.

Suppose that $f \in W(\mathbf{R},\mathcal{U})$ and $f_j \in W(\mathbf{R},\mathcal{U}_j)$, $j \in \mathbf{N}$, where the $\mathcal{U}_j$ are (complete) metric spaces. Then $\mathrm{Mod}\, f \subset \sum_j \mathrm{Mod}\, f_j$ if and only if every sequence $\tau_k \in \mathbf{R}$, $k \in \mathbf{N}$, that is f_j-returning for all $j \in \mathbf{N}$ is f-returning. In particular, for the case when $f_j \in W(\mathbf{R},\mathcal{U}_j)$, $j = 1, 2$, we have $\mathrm{Mod}\, f_1 \subset \mathrm{Mod}\, f_2$ if and only if every f_2-returning sequence $\{\tau_k\}$ is f_1-returning.

Lemma 2.4. *Suppose that $f \in M_p(\mathbf{R},\mathcal{U})$, $f_j \in W_p(\mathbf{R},\mathcal{U})$, $j \in \mathbf{N}$, and $D_p^{(\rho),W}(f, f_j) \to 0$ as $j \to +\infty$. Then $f \in W_p(\mathbf{R},\mathcal{U})$ and $\mathrm{Mod}\, f \subset \sum_j \mathrm{Mod}\, f_j$.*

Corollary 2.1. *Suppose that* $f \in M(\mathbf{R},\mathcal{U})$, $f_j \in W(\mathbf{R},\mathcal{U})$, $j \in \mathbf{N}$, *and* $D^{(\rho),W}(f,f_j) \to 0$ *as* $j \to +\infty$. *Then* $f \in W(\mathbf{R},\mathcal{U})$ *and* $\operatorname{Mod} f \subset \sum_j \operatorname{Mod} f_j$.

Theorem 2.2 (see e.g. [12]). *Let* $(\mathcal{H}, \|.\|)$ *be a complex Banach space and let* $f \in W_p(\mathbf{R},\mathcal{H})$, $p \geq 1$. *Then for any* $\epsilon > 0$ *there is a trigonometric polynomial*

$$f_\epsilon(t) = \sum_{j=1}^{N(\epsilon)} c_j^{(\epsilon)} e^{i\lambda_j^{(\epsilon)} t}, \ t \in \mathbf{R},$$

where $c_j^{(\epsilon)} \in \mathcal{H}$, $\lambda_j^{(\epsilon)} \in \mathbf{R}$ *(and the sum contains only a finite number of terms), such that* $\|f - f_\epsilon\|_p < \epsilon$ *and* $\Lambda\{f_\epsilon\} \subset \Lambda\{f\}$.

The following Theorem is a consequence of Theorem 2.2.

Theorem 2.3. *Let* $f \in W(\mathbf{R},\mathcal{H})$. *Then for any* $\epsilon > 0$ *there is a trigonometric polynomial* $f_\epsilon \in CAP\,(\mathbf{R},\mathcal{H})$ *such that* $D^{(\rho),W}(f,f_\epsilon) < \epsilon$ *and* $\operatorname{Mod} f_\epsilon \subset \operatorname{Mod} f$.

Corollary 2.2. *Let* $f_1 , f_2 \in W(\mathbf{R},\mathcal{H})$, *then* $f_1 + f_2 \in W(\mathbf{R},\mathcal{H})$ *and* $\operatorname{Mod}(f_1 + f_2) \subset \operatorname{Mod} f_1 + \operatorname{Mod} f_2$. *If* $f \in W(\mathbf{R},\mathcal{H})$, $g \in W(\mathbf{R},\mathbf{C})$, *then also* $gf \in W(\mathbf{R},\mathcal{H})$ *and* $\operatorname{Mod} gf \subset \operatorname{Mod} f + \operatorname{Mod} g$.

For a set $T \in W(\mathbf{R})$ we have $\mathbf{R}\backslash T \in W(\mathbf{R})$ and $\operatorname{Mod} \mathbf{R}\backslash T = \operatorname{Mod} T$.

Lemma 2.5. *Let* $T_1 , T_2 \in W(\mathbf{R})$. *Then* $T_1 \bigcup T_2 \in W(\mathbf{R})$, $T_1 \bigcap T_2 \in W(\mathbf{R})$, $T_1\backslash T_2 \in W(\mathbf{R})$ *and modules* $\operatorname{Mod} T_1 \bigcup T_2$, $\operatorname{Mod} T_1 \bigcap T_2$, $\operatorname{Mod} T_1\backslash T_2$ *are subsets (subgroups) of* $\operatorname{Mod} T_1 + \operatorname{Mod} T_2$.

Corollary 2.3. *Let* Λ *be a module in* $\mathbf{R}$ *and let* $\{T_j^{(s)}\}_{j\in\mathbf{N}} \in \mathfrak{M}^{(W)}(\Lambda)$, $s = 1, 2$, *then also* $\{T_j^{(1)} \bigcap T_k^{(2)}\}_{j,k\in\mathbf{N}} \in \mathfrak{M}^{(W)}(\Lambda)$.

Proof of Theorem 1.1. By the Fréchet Theorem (on isometric embedding of a metric space into some Banach space) [13], we can suppose that $\mathcal{U} = (\mathcal{H}, \|.\|)$. From Corollary 2.2 it follows that for all $N \in \mathbf{N}$

$$\sum_{j=1}^{N} f_j(.)\chi_{T_j}(.) \in W(\mathbf{R},\mathcal{H})$$

and

$$\operatorname{Mod} \sum_{j=1}^{N} f_j(.)\chi_{T_j}(.) \subset \sum_{j=1}^{N} \operatorname{Mod} f_j + \sum_{j=1}^{N} \operatorname{Mod} T_j \,.$$

On the other hand, we have

$$D^{(\rho),W}\big(\sum_{j=1}^{+\infty} f_j(.)\chi_{T_j}(.)\,, \sum_{j=1}^{N} f_j(.)\chi_{T_j}(.)\big) \to 0$$

as $N \to +\infty$. To complete the proof it remains to apply Corollary 2.1. $\square$

3 Proof of Theorem 1.2

For $h \in (\mathcal{H}, \|.\|)$ we set

$$\operatorname{sgn} h = \begin{cases} \frac{h}{\|h\|} & \text{if } h \neq 0\,, \\ 0 & \text{if } h = 0\,. \end{cases}$$

Lemma 3.1. *Let $f \in W(\mathbf{R}, \mathcal{H})$. Suppose that for any $\epsilon > 0$ there are numbers $\delta > 0$ and $l > 0$ such that*

$$\sup_{\xi \in \mathbf{R}} \operatorname{meas}\{t \in [\xi, \xi + l] : \|f(t)\| < \delta\} < \epsilon l\,. \tag{2}$$

Then $\operatorname{sgn} f(.) \in W_1(\mathbf{R}, \mathcal{H})$ *and* $\operatorname{Mod} \operatorname{sgn} f(.) \subset \operatorname{Mod} f(.)$ *(furthermore, for the set $T = \{t \in \mathbf{R} : f(t) = 0\}$ we have $\|\chi_T(.)\|_1 = 0$).*

Proof. For all $j \in \mathbf{N}$ define the functions

$$\mathcal{H} \ni h \to \mathcal{F}_j(h) = \begin{cases} jh & \text{if } \|h\| \leq \frac{1}{j}\,, \\ \frac{h}{\|h\|} & \text{if } \|h\| > \frac{1}{j}\,. \end{cases}$$

These functions are uniformly continuous and bounded, therefore $\mathcal{F}_j(f(.)) \in W_1(\mathbf{R}, \mathcal{H})$ and $\operatorname{Mod} \mathcal{F}_j(f(.)) \subset \operatorname{Mod} f(.)$. From the condition (2) it follows that $\|\chi_T(.)\|_1 = 0$ and $\|\operatorname{sgn} f(.) - \mathcal{F}_j(f(.))\|_1 \to 0$ as $j \to +\infty$. Hence (in view of Lemma 2.4) $\operatorname{sgn} f(.) \in W_1(\mathbf{R}, \mathcal{H})$ and $\operatorname{Mod} \operatorname{sgn} f(.) \subset \sum_j \operatorname{Mod} \mathcal{F}_j(f(.)) \subset \operatorname{Mod} f(.)$. □

Lemma 3.2. *Let $f \in W(\mathbf{R}, \mathcal{U})$. Then for any $\epsilon, \delta > 0$ there exist a number $l > 0$ and finitely many points $x_j \in \mathcal{U}$, $j = 1, \dots, N$, such that*

$$\sup_{\xi \in \mathbf{R}} \operatorname{meas}\{t \in [\xi, \xi + l] : \rho(f(t), \bigcup_{j=1}^{N} x_j) \geq \delta\} < \epsilon l\,.$$

Proof. Lemma 2.1 implies that for any $\epsilon, \delta > 0$ there exist a number $l > 0$ and a relatively dense set $T \subset \mathbf{R}$ such that the inequality

$$\sup_{\xi \in \mathbf{R}} \operatorname{meas}\{t \in [\xi, \xi + l] : \rho(f(t), f(t + \tau)) \geq \frac{\delta}{2}\} < \frac{\epsilon}{2}\, l$$

holds for all $\tau \in T$. We choose a number $a \geq \frac{l}{2}$ such that for any $\xi \in \mathbf{R}$ there is a number $\tau(\xi) \in T$ for which $[\xi + \tau(\xi), \xi + l + \tau(\xi)] \subset [-a, a]$. Since the function f is measurable, we have

$$\operatorname{meas}\{t \in [-a, a] : \rho(f(t), \bigcup_{j=1}^{N} x_j) \geq \frac{\delta}{2}\} < \frac{\epsilon}{2}\, l$$

for some finite set of points $x_j \in \mathcal{U}$, $j = 1, \ldots, N$, and hence

$$\sup_{\xi \in \mathbf{R}} \text{meas}\,\{t \in [\xi, \xi + l] : \rho(f(t), \bigcup_{j=1}^{N} x_j) \geq \delta\} \leq$$

$$\leq \sup_{\xi \in \mathbf{R}} \Big(\text{meas}\,\{t \in [\xi, \xi + l] : \rho(f(t), f(t + \tau(\xi))) \geq \frac{\delta}{2}\} +$$

$$+ \text{meas}\,\{t \in [\xi, \xi + l] : \rho(f(t + \tau(\xi))), \bigcup_{j=1}^{N} x_j) \geq \frac{\delta}{2}\} \Big) \leq$$

$$\leq \frac{\epsilon}{2}\, l + \text{meas}\,\{t \in [-a, a] : \rho(f(t), \bigcup_{j=1}^{N} x_j) \geq \frac{\delta}{2}\} < \frac{\epsilon}{2}\, l + \frac{\epsilon}{2}\, l = \epsilon l\,.$$

$\square$

Corollary 3.1. *Let $f \in W(\mathbf{R}, \mathcal{U})$. Then for any $\delta > 0$ there are points $x_j \in \mathcal{U}$, $j \in \mathbf{N}$, such that*

(1) for all $a > 0$

$$\lim_{N \to +\infty} \text{meas}\,\{t \in [-a, a] : \rho(f(t), \bigcup_{j \leq N} x_j) \geq \delta\} = 0\,,$$

(2) for any $\epsilon > 0$ there exist numbers $l > 0$ and $N \in \mathbf{N}$ such that

$$\sup_{\xi \in \mathbf{R}} \text{meas}\,\{t \in [\xi, \xi + l] : \rho(f(t), \bigcup_{j \leq N} x_j) \geq \delta\} < \epsilon l\,.$$

Let $\mathcal{A}^{(W)}$ be the collection of sets $\mathbb{F} \subset W(\mathbf{R}, \mathbf{R})$ such that for any $\epsilon > 0$ there exist numbers $l = l(\epsilon, \mathbb{F}) > 0$ and $\tau_0 = \tau_0(\epsilon, \mathbb{F}) > 0$ for which

$$\sup_{f \in \mathbb{F}} \sup_{\tau \in [0, \tau_0]} D_l^{(\rho)}(f(.), f(. + \tau)) < \epsilon$$

$(\rho(x, y) = |x - y|$, $x, y \in \mathbf{R})$.

Lemma 3.3 ([12]). *Let $f \in W_p(\mathbf{R}, \mathcal{U})$. Then for any $\epsilon > 0$ there are numbers $l > 0$ and $\tau_0 > 0$ such that the inequality $D_{p,\,l}^{(\rho)}(f(.), f(. + \tau)) < \epsilon$ holds for all $\tau \in [0, \tau_0]$.*

From Lemma 3.3 it follows that $\{f\} \in \mathcal{A}^{(W)}$ for any function $f \in W(\mathbf{R}, \mathbf{R})$.
The following Theorem is proved in Section 4 and its special case for the set $\mathbb{F} = \{f\}$, $f \in W(\mathbf{R}, \mathbf{R})$, is essentially used in the proof of Theorem 1.2.

Theorem 3.1. *Let $\mathbb{F} \in \mathcal{A}^{(W)}$, $\Delta > 0$, $b > 0$, $\epsilon \in (0,1]$. Then there exist b-periodic function $g(.) \in C(\mathbf{R},\mathbf{R})$ dependent on $\mathbb{F}$, Δ, b, but not on the number ϵ, for which $\|g\|_{L_\infty(\mathbf{R},\mathbf{R})} < \Delta$, and numbers $\delta = \delta(\epsilon,\Delta) > 0$, $l = l(\epsilon,\Delta,\mathbb{F}) > 0$ such that for all functions $f \in \mathbb{F}$*

$$\sup_{\xi\in\mathbf{R}} \operatorname{meas}\{t \in [\xi,\xi+l] : |f(t)+g(t)| < \delta\} < \epsilon l\,.$$

Corollary 3.2. *Let $f \in W(\mathbf{R},\mathbf{R})$. Then for any $a \in \mathbf{R}$ and $\epsilon > 0$ there is a set $T \in W(\mathbf{R})$ such that $\operatorname{Mod} T \subset \operatorname{Mod} f$, $f(t) < a+\epsilon$ for all $t \in T$ and $f(t) > a$ for a.e. $t \in \mathbf{R}\backslash T$.*

Proof of Theorem 1.2. If $\operatorname{Mod} f = \{0\}$, then for some constant function $f_0(t) \equiv f_0 \in \mathcal{U}$, $t \in \mathbf{R}$, we have $D^{(\rho),W}(f(.),f_0(.)) = 0$, and therefore, there is a set $T \in W(\mathbf{R})$ such that $\|\chi_T(.)\|_1 = 0$ and $\rho(f(t),f_0) < \epsilon$ for all $t \in \mathbf{R}\backslash T$. Next, suppose that $\operatorname{Mod} f \neq \{0\}$. Let $x_j \in \mathcal{U}$, $j \in \mathbf{N}$, be the points defined in Corollary 3.1 for the function $f \in W(\mathbf{R},\mathcal{U})$ and the number $\delta = \frac{\epsilon}{3}$. For all $j \in \mathbf{N}$ we have $\rho(f(.),x_j) \in W(\mathbf{R},\mathbf{R})$ and $\operatorname{Mod}\rho(f(.),x_j) \subset \operatorname{Mod} f(.)$. We choose a number $b > 0$ such that $\frac{2\pi}{b} \in \operatorname{Mod} f$. Theorem 3.1 implies the existence of b-periodic function $g_j(.) \in C(\mathbf{R},\mathbf{R})$, $j \in \mathbf{N}$, such that $\|g_j\|_{L_\infty(\mathbf{R},\mathbf{R})} < \frac{\epsilon}{3}$ and for any $\epsilon' \in (0,1]$ there exist numbers $\delta_j = \delta_j(\epsilon',\epsilon) > 0$ and $l_j = l_j(\epsilon',\epsilon,f) > 0$ for which

$$\sup_{\xi\in\mathbf{R}} \operatorname{meas}\{t \in [\xi,\xi+l_j] : |\rho(f(t),x_j) - \frac{2\epsilon}{3} + g_j(t)| < \delta_j\} < \epsilon' l_j\,.$$

Let $T'_j = \{\, t \in \mathbf{R} : \rho(f(t),x_j) + g_j(t) \leq \frac{2\epsilon}{3}\}$, $j \in \mathbf{N}$. According to Lemma 3.1, we get $T'_j \in W(\mathbf{R})$ and $\operatorname{Mod} T'_j \subset \operatorname{Mod}\rho(f(.),x_j) + \frac{2\pi}{b}\mathbf{Z} \subset \operatorname{Mod} f(.)$. If $t \in T'_j$, then $\rho(f(t),x_j) < \epsilon$. We denote $T_1 = T'_1$ and $T_j = T'_j\backslash\bigcup_{k<j} T'_k$ for $j \geq 2$. The sets T_j, $j \in \mathbf{N}$, are disjoint and $\bigcup_{j\leq N} T_j = \bigcup_{j\leq N} T'_j$ for all $N \in \mathbf{N}$. It follows from Lemma 2.5 that $T_j \in W(\mathbf{R})$, $\operatorname{Mod} T_j \subset \operatorname{Mod} f$. Furthermore, $\rho(f(t),x_j) < \epsilon$ for all $t \in T_j$, $j \in \mathbf{N}$, and for every $N \in \mathbf{N}$ and a.e. $t \in \mathbf{R}\backslash\bigcup_{j\leq N} T_j$ we have $\rho(f(t),x_j) \geq \frac{\epsilon}{3}$ for all $j = 1,\dots,N$. Hence (see Corollary 3.1) $\operatorname{meas}\mathbf{R}\setminus\bigcup_j T_j = 0$ and $\|\chi_{\mathbf{R}\setminus\bigcup_j T_j}(.)\|_1 \to 0$ as $N \to +\infty$, that is, $\{T_j\} \in \mathfrak{M}^{(W)}(\operatorname{Mod} f)$. □

Theorems 1.1 and 1.2 enable us to get the following characterization of functions $W(\mathbf{R},\mathcal{U})$: a function $f : \mathbf{R} \to \mathcal{U}$ belongs to $W(\mathbf{R},\mathcal{U})$ if and only if for any $\epsilon > 0$ there exist a sequence $\{T_j\} \in \mathfrak{M}^{(W)}(\mathbf{R})$ and points $x_j \in \mathcal{U}$, $j \in \mathbf{N}$, such that the inequality $\rho(f(t),x_j) < \epsilon$ holds for all $t \in T_j$, $j \in \mathbf{N}$.

In the following two Lemmas we consider the superposition of equi-Weyl a.p. functions.

Lemma 3.4. *Let $(\mathcal{U},\rho)$ and $(\mathcal{V},\rho_{\mathcal{V}})$ be complete metric spaces, let $\mathcal{F} \in C(\mathcal{U},\mathcal{V})$ and let $f \in W(\mathbf{R},\mathcal{U})$. Then $\mathcal{F}(f(.)) \in W(\mathbf{R},\mathcal{V})$ and $\operatorname{Mod}\mathcal{F}(f(.)) \subset \operatorname{Mod} f(.)$.*

Proof. Let $\epsilon \in (0,1]$, $\delta > 0$. By Theorem 1.2, for every $k \in \mathbf{N}$ there are sequences $\{T_j^{(k)}\} \in \mathfrak{M}^{(W)}(\operatorname{Mod} f)$ and points $x_j^{(k)} \in \mathcal{U}$, $j \in \mathbf{N}$, such that $\rho(f(t), x_j^{(k)}) < \frac{1}{k}$ for all $t \in T_j^{(k)}$, $j \in \mathbf{N}$. The inequalities

$$\|\chi_{\mathbf{R} \setminus \bigcup\limits_{j=1}^{j(k)} T_j^{(k)}}(.)\|_1 < 2^{-k}\epsilon\,.$$

hold for some numbers $j(k) \in \mathbf{N}$, $k \in \mathbf{N}$. For every $k \in \mathbf{N}$ we denote by X_k the set of points $x_j^{(k)}$, $j = 1, \ldots, j(k)$, for which for any $k' = 1, \ldots, k-1$ there exists a point $x_{j'}^{(k')}$, $j' = 1, \ldots, j(k')$, such that $\rho(x_j^{(k)}, x_{j'}^{(k')}) < \frac{1}{k} + \frac{1}{k'} < \frac{2}{k'}$; $X_1 = \bigcup_{j \leq j(1)} x_j^{(1)}$. Since the set $\bigcup_{k \in \mathbf{N}} X_k \subset \mathcal{U}$ is precompact and the function $\mathcal{F}$ is continuous, it follows that there is a number $k_0 \in \mathbf{N}$ such that for all $j_k = 1, \ldots, j(k)$, where $k = 1, \ldots, k_0$, the inequality $\rho_{\mathcal{V}}(\mathcal{F}(f(t)), \mathcal{F}(f(t'))) < \delta$ holds for all $t, t' \in T_{j_1}^{(1)} \bigcap \cdots \bigcap T_{j_{k_0}}^{(k_0)}$. If $T_{j_1}^{(1)} \bigcap \cdots \bigcap T_{j_{k_0}}^{(k_0)} \neq \emptyset$, where $j_k = 1, \ldots, j(k)$, $k = 1, \ldots, k_0$, we choose some numbers $t_{j_1 \ldots j_{k_0}} \in T_{j_1}^{(1)} \bigcap \cdots \bigcap T_{j_{k_0}}^{(k_0)}$. Let

$$T(k_0) = \bigcap_{k=1,\ldots,k_0} \bigcup_{j_k=1}^{j(k)} T_{j_k}^{(k)}\,.$$

By Lemma 2.5 and Theorem 1.1,

$$\mathcal{G}_{k_0}(.) \doteq \sum_{j_k=1,\ldots,j(k);\, k=1,\ldots,k_0} \mathcal{F}(f(t_{j_1 \ldots j_{k_0}}))\chi_{T_{j_1}^{(1)} \bigcap \cdots \bigcap T_{j_{k_0}}^{(k_0)}}(.) + y_0 \chi_{\mathbf{R} \setminus T(k_0)}(.) \in W(\mathbf{R}, \mathcal{V})\,,$$

where $y_0 \in \mathcal{V}$, and

$$\operatorname{Mod} \mathcal{G}_{k_0}(.) \subset \sum_{j_k=1,\ldots,j(k);\, k=1,\ldots,k_0} \operatorname{Mod} T_{j_k}^{(k)} \subset \operatorname{Mod} f(.)\,.$$

Furthermore, $\rho_{\mathcal{V}}(\mathcal{F}(f(t)), \mathcal{G}_{k_0}(t)) < \delta$ for all $t \in T(k_0)$ and

$$\|\chi_{\mathbf{R} \setminus T(k_0)}(.)\|_1 \leq \sum_{k=1,\ldots,k_0} 2^{-k}\epsilon < \epsilon\,.$$

Hence $D^{(\rho_{\mathcal{V}}),W}(\mathcal{F}(f(.)), \mathcal{G}_{k_0}(.)) < \epsilon + \delta$. Since the numbers $\epsilon > 0$ and $\delta > 0$ can be chosen arbitraryly small, it follows from Corollary 2.1 that $\mathcal{F}(f(.)) \in W(\mathbf{R}, \mathcal{V})$ and $\operatorname{Mod} \mathcal{F}(f(.)) \subset \operatorname{Mod} f(.)$. $\square$

On the set $C(\mathcal{U}, \mathcal{V})$, where $(\mathcal{U}, \rho)$ and $(\mathcal{V}, \rho_{\mathcal{V}})$ are metric spaces, we introduce the metric

$$d_{C(\mathcal{U},\mathcal{V})}(\mathcal{F}_1, \mathcal{F}_2) = \sup_{x \in \mathcal{U}} \min\{1, \rho_{\mathcal{V}}(\mathcal{F}_1(x), \mathcal{F}_2(x))\}\,,\ \mathcal{F}_1\,,\ \mathcal{F}_2 \in C(\mathcal{U}, \mathcal{V})\,.$$

Lemma 3.5. *Let $(\mathcal{U},\rho)$ and $(\mathcal{V},\rho_{\mathcal{V}})$ be complete metric spaces. Suppose that a function $\mathbf{R}\ni t\to \mathcal{F}(.;t)\in C(\mathcal{U},\mathcal{V})$ belongs to the space $W_1(\mathbf{R},(C(\mathcal{U},\mathcal{V}),d_{C(\mathcal{U},\mathcal{V})}))$ and $f\in W(\mathbf{R},\mathcal{U})$. Then $\mathcal{F}(f(.);.)\in W(\mathbf{R},\mathcal{V})$ and* $\operatorname{Mod}\mathcal{F}(f(.);.)\subset \operatorname{Mod}\mathcal{F}(.;.)+\operatorname{Mod} f(.)$.

Proof. Theorem 1.2 implies that for any $\epsilon>0$ there are a sequence $\{T_j\}\in \mathfrak{M}^{(W)}(\operatorname{Mod}\mathcal{F}(.;.))$ and functions $\mathcal{F}_j\in C(\mathcal{U},\mathcal{V})$, $j\in\mathbf{N}$, such that $d_{C(\mathcal{U},\mathcal{V})}(\mathcal{F}(.;t),\mathcal{F}_j(.))<\epsilon$ for all $t\in T_j$, $j\in\mathbf{N}$. By Theorem 1.1 and Lemma 3.4,

$$\sum_{j\in\mathbf{N}}\mathcal{F}_j(f(.))\chi_{T_j}(.)\in W(\mathbf{R},\mathcal{V})\,,$$

$$\operatorname{Mod}\sum_{j\in\mathbf{N}}\mathcal{F}_j(f(.))\chi_{T_j}(.)\subset \operatorname{Mod}\mathcal{F}(.;.)+\operatorname{Mod} f(.)\,.$$

On the other hand,

$$D^{(\rho_{\mathcal{V}}),W}\big(\mathcal{F}(f(.);.),\sum_{j\in\mathbf{N}}\mathcal{F}_j(f(.))\chi_{T_j}(.)\big)<\epsilon\,.$$

Hence, by Corollary 2.1, we get $\mathcal{F}(f(.);.)\in W(\mathbf{R},\mathcal{V})$ and $\operatorname{Mod}\mathcal{F}(f(.);.)\subset \operatorname{Mod}\mathcal{F}(.;.)+\operatorname{Mod} f(.)$. □

Remark 2. From Lemma 3.4, Theorems 1.1, 1.2 and 2.1 we obtain also the following assertion. Let $(\mathcal{U},\rho)$ and $(\mathcal{V},\rho_{\mathcal{V}})$ be complete metric spaces, let $r>0$ and let $p\geq 1$. Suppose that a function $\mathbf{R}\ni t\to\mathcal{F}(.;t)\in C(\mathcal{U},\mathcal{V})$ satisfies the following two conditions:

(1) for any $x\in\mathcal{U}$ the function $\mathbf{R}\ni t\to \mathcal{F}(.|_{B_r(x)};t)\in C(B_r(x),\mathcal{V})$ belongs to the space

$$W_1(\mathbf{R},(C(B_r(x),\mathcal{V}),d_{C(B_r(x),\mathcal{V})}))$$

(we denote by $\mathcal{F}(.|_Y)$ the restriction of a function $\mathcal{F}(.)\in C(\mathcal{U},\mathcal{V})$ to a non-empty set $Y\subset\mathcal{U}$);

(2) for a.e. $t\in\mathbf{R}$ the inequality

$$\rho_{\mathcal{V}}(\mathcal{F}(x;t),y_0)\leq A\rho(x,x_0)+B(t)$$

holds for all $x\in\mathcal{U}$, where $x_0\in\mathcal{U}$ and $y_0\in\mathcal{V}$ are some fixed points, $A\geq 0$, $B(.)\in M^{\sharp}_p(\mathbf{R},\mathbf{R})$.

Then for any function $f\in W_p(\mathbf{R},\mathcal{U})$ we have $\mathcal{F}(f(.);.)\in W_p(\mathbf{R},\mathcal{V})$ and

$$\operatorname{Mod}\mathcal{F}(f(.);.)\subset \operatorname{Mod} f(.)+\sum_{x\in\mathcal{U}}\operatorname{Mod}\mathcal{F}(.|_{B_r(x)};.)\,.$$

4 Proof of Theorem 3.1

Lemma 4.1. *Let* $\mathbb{F} \in \mathcal{A}^{(W)}$, $\Delta > 0$. *Then for any* $\epsilon \in (0,1]$ *there exist numbers* $\delta = \delta(\epsilon,\Delta) > 0$, $l = l(\epsilon,\Delta,\mathbb{F}) > 0$ *and* $\widetilde{\alpha} = \widetilde{\alpha}(\epsilon,\Delta,\mathbb{F}) > 0$ *such that for all* $\alpha \geq \widetilde{\alpha}$ *and all functions* $f \in \mathbb{F}$

$$\sup_{\xi\in\mathbf{R}} \operatorname{meas}\{t \in [\xi,\xi+l] : |f(t)+\Delta\sin\alpha t| < \delta\} < \epsilon l\,.$$

Proof. Let us choose a number $N = N(\epsilon) \in \mathbf{N}$ for which $(N+1)^{-1} < \frac{\epsilon}{3}$ (then $N \geq 3$). Let $\epsilon' \doteq \frac{1}{3}\epsilon N^{-1}(N+1)^{-1} \leq \frac{\epsilon}{36} < 1$, $\delta' = 2\sin\frac{\pi}{2N}\sin\frac{\pi\epsilon'}{6}$, $\delta = \delta(\epsilon,\Delta) = \min\{1, \frac{1}{3}\delta'\Delta\}$. There are numbers $l = l(\epsilon,\Delta,\mathbb{F}) > 0$ and $\tau_0 = \tau_0(\epsilon,\Delta,\mathbb{F}) \in (0,\frac{2}{9}\epsilon] \subset (0,1)$ such that the inequality

$$D_l^{(\rho)}(f(.), f(.+\tau)) < \epsilon'\delta$$

holds for all $f \in \mathbb{F}$ and $\tau \in [0,\tau_0]$. We define the number $\widetilde{\alpha} = \pi\tau_0^{-1}$. Let $0 < \tau \leq \tau_0$, $\alpha \doteq \pi\tau^{-1} \geq \widetilde{\alpha}$. For $j = 1,\dots,N$ (and $f \in \mathbb{F}$, $\xi \in \mathbf{R}$) let us define the sets

$$\mathcal{L}_j(\xi) = \mathcal{L}_j(f,\tau;\xi) = \{t \in [\xi,\xi+l] : |f(t+\frac{j}{N}\tau) - f(t)| \geq \delta\}\,.$$

We have

$$\operatorname{meas}\mathcal{L}_j(\xi) \leq \frac{1}{\delta}\int_\xi^{\xi+l} \min\{1, |f(t+\frac{j}{N}\tau) - f(t)|\}\,dt < \epsilon' l\,.$$

For $j = 1,\dots,N$ (and $\xi \in \mathbf{R}$) we also consider the sets

$$\mathcal{M}_j(\xi) = \mathcal{M}_j(\tau;\xi) = \{t \in [\xi,\xi+l] : |\cos\alpha(t+\frac{j}{2N}\tau)\,\sin\frac{\alpha j}{2N}\tau| \leq \frac{\delta'}{2}\}\,,$$

$$\mathcal{M}'_j(\xi) = \mathcal{M}'_j(\tau;\xi) = \{t \in [\xi,\xi+l] : |\cos(\alpha t+\frac{j\pi}{2N})| \leq \frac{1}{2}\delta'\sin^{-1}\frac{\pi}{2N} = \sin\frac{\pi\epsilon'}{6}\}\,;$$

$\mathcal{M}_j(\xi) \subset \mathcal{M}'_j(\xi)$ and $\mathcal{M}'_j(\xi) = [\xi,\xi+l] \bigcap \left(\bigcup_{s\in\mathbf{Z}} [\beta_s^-, \beta_s^+]\right)$, where

$$\beta_s^{\pm} = \left(s+\frac{1}{2}\right)\frac{\pi}{\alpha} - \frac{j\pi}{2N\alpha} \pm \frac{\pi\epsilon'}{6\alpha}\,.$$

Let κ be the number of closed intervals $[\beta_s^-, \beta_s^+]$, $s \in \mathbf{Z}$, intersecting with the closed interval $[\xi,\xi+l]$. We have the estimate $\kappa \leq (\frac{\pi}{\alpha})^{-1}(l+\frac{2\pi}{\alpha}) = \frac{l\alpha}{\pi}+2$, hence

$$\operatorname{meas}\mathcal{M}_j(\xi) \leq \operatorname{meas}\mathcal{M}'_j(\xi) \leq \kappa\frac{\pi\epsilon'}{3\alpha} \leq \left(\frac{l}{3}+\frac{2}{3}\tau_0\right)\epsilon' \leq l\epsilon'\,.$$

In what follows, we suppose that the sets $\mathcal{L}_j(\xi)$ contain (in addition) the numbers $t \in [\xi, \xi + l]$ for which at the least one of the functions $f(t)$, $f(t + \frac{j}{N}\tau)$ is not defined (these numbers form the set of measure zero). Let

$$\mathcal{L}(\xi) = \mathcal{L}(f, \tau; \xi) = \bigcup_{j=1}^{N} \Big(\bigcup_{s=0}^{N-j} (\mathcal{L}_j(\xi) - \frac{s}{N}\tau) \Big)$$

(here $\mathcal{L}_j(\xi) - \frac{s}{N}\tau = \{t = \eta - \frac{s}{N}\tau : \eta \in \mathcal{L}_j(\xi)\}$). Since $\operatorname{meas} \mathcal{L}_j(\xi) < \epsilon' l$, $j = 1, \ldots, N$, we get $\operatorname{meas} \mathcal{L}(\xi) < \frac{1}{2} N(N+1)\epsilon' l = \frac{1}{6}\epsilon l$. If $t \in [\xi, \xi + l - \tau] \backslash \mathcal{L}(\xi)$, then the numbers $t + \frac{j}{N}\tau$, $j = 0, 1, \ldots, N$, belong to the closed interval $[\xi, \xi + l]$ and

$$|f(t + \frac{j_1}{N}\tau) - f(t + \frac{j_2}{N}\tau)| < \delta$$

for all $j_1, j_2 \in \{0, 1, \ldots, N\}$. Let

$$\mathcal{M}(\xi) = \mathcal{M}(\tau; \xi) = \bigcup_{j=1}^{N} \Big(\bigcup_{s=0}^{N-j} (\mathcal{M}_j(\xi) - \frac{s}{N}\tau) \Big);$$

$\operatorname{meas} \mathcal{M}(\xi) \le \frac{1}{2} N(N+1)\epsilon' l = \frac{1}{6}\epsilon l$. If $t \in [\xi, \xi + l - \tau] \backslash \mathcal{M}(\xi)$, then the numbers $t + \frac{j}{N}\tau$, $j = 0, 1, \ldots, N$, also belong to the closed interval $[\xi, \xi + l]$ and for all $j_1, j_2 \in \{0, 1, \ldots, N\}$, $j_1 < j_2$, we have

$$|\Delta \sin \alpha(t + \frac{j_1}{N}\tau) - \Delta \sin \alpha(t + \frac{j_2}{N}\tau)| =$$

$$= 2\Delta |\cos \alpha(t + \frac{j_1}{N}\tau + \frac{j_2 - j_1}{2N}\tau) \sin \alpha \frac{j_2 - j_1}{2N}\tau| > \Delta\delta' \ge 3\delta .$$

Let $G(t) = f(t) + \Delta \sin \alpha t$, $t \in \mathbf{R}$. We define (for $\xi \in \mathbf{R}$) the set

$$\mathcal{O}(\xi) = \mathcal{O}(f, \tau; \xi) = [\xi, \xi + l - \tau] \backslash (\mathcal{L}(\xi) \bigcup \mathcal{M}(\xi)) .$$

For each $t \in \mathcal{O}(\xi)$ either $|G(t + \frac{j}{N}\tau)| \ge \delta$ for all $j = 0, 1, \ldots, N$ or there exists a number $j_0 \in \{0, 1, \ldots, N\}$ such that $|G(t + \frac{j_0}{N}\tau)| < \delta$. Consider the minimal number j_0 for which the last inequality holds. If $j_0 < N$, then for any $j \in \{j_0 + 1, \ldots, N\}$ we have

$$|G(t + \frac{j}{N}\tau) - G(t + \frac{j_0}{N}\tau)| \ge$$

$$\ge |\Delta \sin \alpha(t + \frac{j}{N}\tau) - \Delta \sin \alpha(t + \frac{j_0}{N}\tau)| - |f(t + \frac{j}{N}\tau) - f(t + \frac{j_0}{N}\tau)| > 3\delta - \delta = 2\delta ,$$

and therefore, $|G(t + \frac{j}{N}\tau)| > \delta$. We have got that in the case $t \in \mathcal{O}(\xi)$ there is at most one number $t + \frac{j}{N}\tau$, $j = 0, 1, \ldots, N$, such that $|G(t + \frac{j}{N}\tau)| < \delta$. Let

denote by $\chi(t)$, $t \in \mathbf{R}$, the characteristic function of the set $\{t \in [\xi, \xi + l - \tau] : |G(t)| < \delta\}$;

$$\widetilde{\chi}(t) = \sum_{j=0}^{N} \chi(t + \frac{j}{N}\tau)\,,\ t \in \mathbf{R}\,.$$

Then $\widetilde{\chi}(t) \le 1$ for all $t \in \mathcal{O}(\xi)$, and hence

$$\int\limits_{\xi}^{\xi+l-\tau} \widetilde{\chi}(t)\,dt = \int\limits_{\mathcal{O}(\xi)} \widetilde{\chi}(t)\,dt + \int\limits_{[\xi,\xi+l-\tau]\setminus\mathcal{O}(\xi)} \widetilde{\chi}(t)\,dt \le$$

$$\le \operatorname{meas}\mathcal{O}(\xi) + (N+1)\operatorname{meas}(\mathcal{L}(\xi) \bigcup \mathcal{M}(\xi)) < \big(1 + \frac{1}{3}(N+1)\epsilon\big)l\,.$$

On the other hand,

$$\int\limits_{\xi}^{\xi+l-\tau} \widetilde{\chi}(t)\,dt = (N+1)\int\limits_{\xi}^{\xi+l-\tau} \chi(t)\,dt - \sum_{j=1}^{N} \int\limits_{\xi}^{\xi+\frac{j}{N}\tau} \chi(t)\,dt \ge$$

$$\ge (N+1)\operatorname{meas}\{t \in [\xi, \xi+l-\tau] : |G(t)| < \delta\} - \frac{1}{2}(N+1)\tau \ge$$

$$\ge (N+1)\operatorname{meas}\{t \in [\xi, \xi+l] : |G(t)| < \delta\} - \frac{3}{2}(N+1)\tau\,.$$

Hence

$$\operatorname{meas}\{t \in [\xi, \xi+l] : |G(t)| < \delta\} <$$

$$< \left(\frac{1}{N+1} + \frac{3\tau}{2l} + \frac{\epsilon}{3}\right)l \le \left(\epsilon - \left(\frac{\epsilon}{3} - \frac{1}{N+1}\right)\right)l < \epsilon l$$

(for all $\xi \in \mathbf{R}$). □

The following Lemma is an immediate consequence of Lemma 4.1.

Lemma 4.2. *Let $\mathbb{F} \in \mathcal{A}^{(W)}$, $\Delta > 0$. Then for any $\epsilon \in (0,1]$ there exist numbers $\delta = \delta(\epsilon, \Delta) > 0$, $l = l(\epsilon, \Delta, \mathbb{F}) > 0$ and $\widetilde{\alpha} = \widetilde{\alpha}(\epsilon, \Delta, \mathbb{F}) > 0$ such that for each function $g \in L_\infty(\mathbf{R}, \mathbf{R})$ satisfying the condition $\|g\|_{L_\infty(\mathbf{R},\mathbf{R})} \le \delta$, and for all $\alpha \ge \widetilde{\alpha}$ and all functions $f \in \mathbb{F}$*

$$\sup_{\xi \in \mathbf{R}} \operatorname{meas}\{t \in [\xi, \xi+l] : |f(t) + \Delta \sin \alpha t + g(t)| < \delta\} < \epsilon l\,.$$

Proof of Theorem 3.1. Let $\Delta_0 = \frac{\Delta}{2}$, $f_0(.) = f(.)$ (for all functions $f \in \mathbb{F}$). By Lemma 4.2, there are numbers $\delta_0 = \delta_0(\Delta) > 0$, $l_0 = l_0(\Delta, \mathbb{F}) > 0$ and $\alpha_0 = \alpha_0(b, \Delta, \mathbb{F}) \in \frac{2\pi}{b}\mathbf{N}$ such that for all functions $f_1(t) \doteq f_0(t) + \Delta_0 \sin \alpha_0 t$, $t \in \mathbf{R}$,

and all functions $\widetilde{g}_1 \in L_\infty(\mathbf{R},\mathbf{R})$ that satisfy the condition $\|\widetilde{g}_1\|_{L_\infty(\mathbf{R},\mathbf{R})} \le \delta_0$, the inequality

$$\sup_{\xi\in\mathbf{R}} \text{meas}\,\{t \in [\xi, \xi + l_0] : |f_1(t) + \widetilde{g}_1(t)| < \delta_0\} < 2^{-1} l_0 \tag{3}$$

holds, furthermore $\{f_1(.) : f \in \mathbb{F}\} \in \mathcal{A}^{(W)}$. We shall successively for $j = 1, 2, \ldots$ find numbers $\Delta_j = \Delta_j(\Delta) > 0$, $\delta_j = \delta_j(\Delta) > 0$, $l_j = l_j(\Delta, \mathbb{F}) > 0$, $\alpha_j = \alpha_j(b, \Delta, \mathbb{F}) \in \frac{2\pi}{b}\mathbf{N}$ and functions $f_{j+1} \in W(\mathbf{R},\mathbf{R})$ dependent on f_j, Δ_j and α_j, for which $\{f_{j+1}(.) : f \in \mathbb{F}\} \in \mathcal{A}^{(W)}$. If the numbers Δ_k, δ_k, l_k, α_k and the functions f_{k+1} have been found for all $k = 0, \ldots, j-1$, where $j \in \mathbf{N}$, then we choose the number $\Delta_j = \Delta_j(\Delta) > 0$ such that the inequalities $\Delta_j < 2^{-(j+1)}\Delta$, $\Delta_j \le 2^{-j}\delta_0$, $\Delta_j \le 2^{-(j-1)}\delta_1$, $\ldots$, $\Delta_j \le 2^{-1}\delta_{j-1}$ hold. Further (according to Lemma 4.2), choose numbers $\delta_j = \delta_j(\Delta) > 0$, $l_j = l_j(\Delta, \mathbb{F}) > 0$ and $\alpha_j = \alpha_j(b, \Delta, \mathbb{F}) \in \frac{2\pi}{b}\mathbf{N}$ such that for all functions $f_{j+1}(t) \doteq f_j(t) + \Delta_j \sin\alpha_j t$, $t \in \mathbf{R}$, and all functions $\widetilde{g}_{j+1} \in L_\infty(\mathbf{R},\mathbf{R})$ satisfying the condition $\|\widetilde{g}_{j+1}\|_{L_\infty(\mathbf{R},\mathbf{R})} \le \delta_j$, the inequality

$$\sup_{\xi\in\mathbf{R}} \text{meas}\,\{t \in [\xi, \xi + l_j] : |f_{j+1}(t) + \widetilde{g}_{j+1}(t)| < \delta_j\} < 2^{-j-1} l_j \tag{4}$$

holds. We also have $\{f_{j+1}(.) : f \in \mathbb{F}\} \in \mathcal{A}^{(W)}$. Next, let us set

$$g(t) = \sum_{j=0}^{+\infty} \Delta_j \sin\,\alpha_j t\,,\ t \in \mathbf{R}\,.$$

Since $\Delta_0 = \frac{\Delta}{2}$ and $\Delta_j < 2^{-(j+1)}\Delta$ for all $j \in \mathbf{N}$, it follows that the function $g(.)$ is continuous and b-periodic, moreover,

$$\|g\|_{L_\infty(\mathbf{R},\mathbf{R})} \le \sum_{j=0}^{+\infty} \Delta_j < \Delta\,.$$

We define the functions

$$g_j(t) = \sum_{k=j}^{+\infty} \Delta_k \sin\,\alpha_k t\,,\ t \in \mathbf{R}\,,\ j \in \mathbf{N}\,.$$

For all $t \in \mathbf{R}$ we have

$$|g_j(t)| \le \sum_{k=j}^{+\infty} \Delta_k \le \Big(\frac{1}{2} + \frac{1}{4} + \ldots\Big)\,\delta_{j-1} = \delta_{j-1}\,,$$

hence, it follows from (3) and (4) that for all numbers $j = 0, 1, \ldots$ (and all functions $f \in \mathbb{F}$) the inequality

$$\sup_{\xi\in\mathbf{R}} \text{meas}\,\{t \in [\xi, \xi + l_j] : |f(t) + g(t)| < \delta_j\} < 2^{-j-1} l_j\,.$$

holds. The proof of Theorem 3.1 is complete.

5 Proof of Theorem 1.3

Theorem 5.1. *Let $(\mathcal{U},\rho)$ be a complete metric space, let $F \in W(\mathbf{R}, \mathrm{cl}\,\mathcal{U})$ and let $g \in W(\mathbf{R},\mathcal{U})$. Then for any $\epsilon > 0$ there exists a function $f \in W(\mathbf{R},\mathcal{U})$ such that* $\mathrm{Mod}\, f \subset \mathrm{Mod}\, F + \mathrm{Mod}\, g$, $f(t) \in F(t)$ *a.e. and* $\rho(f(t), g(t)) < \rho(g(t), F(t)) + \epsilon$ *a.e.*

Proof. Let number $\epsilon \in (0,1]$ be fixed. We choose numbers $\gamma_n > 0$, $n \in \mathbf{N}$, such that

$$\sum_{n=1}^{+\infty} (\gamma_n + \gamma_{n+1}) < \frac{1}{6}$$

(then $\gamma_1 < \frac{1}{6}$). From Theorem 1.2, Lemma 2.5 and Corollary 2.3 it follows that for each $n \in \mathbf{N}$ there exist sets $F_j^{(n)} \in \mathrm{cl}\,\mathcal{U}$, points $g_j^n \in \mathcal{U}$ and disjoint measurable (in the Lebesgue sense) sets $T_j^{(n)} \subset \mathbf{R}$, $j \in \mathbf{N}$, such that $\{T_j^{(n)}\}_{j\in\mathbf{N}} \in \mathfrak{M}^{(W)}(\mathrm{Mod}\, F + \mathrm{Mod}\, g)$, the functions $F(t)$ and $g(t)$ are defined for all $t \in \bigcup_j T_j^{(n)}$, and for all $t \in T_j^{(n)}$, $j \in \mathbf{N}$, we have $\mathrm{dist}_{\rho'}(F(t), F_j^{(n)}) < \gamma_n \epsilon < 1$ and $\rho(g(t), g_j^n) < \gamma_n \epsilon$. Let

$$T = \bigcap_n \bigcup_j T_j^{(n)} ;$$

$\mathrm{meas}\, \mathbf{R}\backslash T = 0$. By Corollary 2.3, for every $n \in \mathbf{N}$

$$\{T_{j_1}^{(1)} \bigcap \cdots \bigcap T_{j_n}^{(n)}\}_{j_s\in\mathbf{N},\ s=1,\dots,n} \in \mathfrak{M}^{(W)}(\mathrm{Mod}\, F + \mathrm{Mod}\, g) .$$

With each number $n \in \mathbf{N}$ and each collection $\{j_1 , \dots , j_n\}$ of indices $j_s \in \mathbf{N}$, $s = 1,\dots,n$, if $T_{j_1}^{(1)} \bigcap \cdots \bigcap T_{j_n}^{(n)} \neq \emptyset$, we associate some point $f_{j_1\dots j_n} \in F_{j_n}^{(n)} \subset \mathcal{U}$. These points are determined successively for $n = 1, 2, \dots$. For $n = 1$ we choose points $f_{j_1} \in F_{j_1}^{(1)}$ such that the inequalities

$$\rho(f_{j_1} , g_{j_1}^1) < \frac{\epsilon}{6} + \rho(g_{j_1}^1 , F_{j_1}^{(1)})$$

hold. If points $f_{j_1\dots j_{n-1}} \in F_{j_{n-1}}^{(n-1)}$ have been found for some $n \geq 2$, then we choose points $f_{j_1\dots j_{n-1}j_n} \in F_{j_n}^{(n)}$ such that

$$\rho(f_{j_1\dots j_{n-1}} , f_{j_1\dots j_{n-1}j_n}) = \rho'(f_{j_1\dots j_{n-1}} , f_{j_1\dots j_{n-1}j_n}) \leq \tag{5}$$

$$\leq 2\, \mathrm{dist}_{\rho'}(F_{j_{n-1}}^{(n-1)}, F_{j_n}^{(n)}) < 2\,(\gamma_{n-1} + \gamma_n)\epsilon < \frac{\epsilon}{3} \leq \frac{1}{3} .$$

Let us define functions

$$f(n;t) = \sum_{j_1,\dots,j_n} f_{j_1\dots j_n} \chi_{T_{j_1}^{(1)} \bigcap \cdots \bigcap T_{j_n}^{(n)}}(t) ,\ t \in T ,\ n \in \mathbf{N} .$$

According to Theorem 1.1, we have $f(n;.) \in W(\mathbf{R},\mathcal{U})$ and $\operatorname{Mod} f(n;.) \subset \operatorname{Mod} F + \operatorname{Mod} g$. It follows from (5) that the inequality

$$\rho(f(n-1;t), f(n;t)) < 2\,(\gamma_{n-1}+\gamma_n)\epsilon \tag{6}$$

holds for all $t \in T$ and $n \geq 2$. Since the metric space $\mathcal{U}$ is complete, we obtain from (6) that the sequence of functions $f(n;.)$, $n \in \mathbf{N}$, converges as $n \to +\infty$ uniformly on the set $T \subset \mathbf{R}$ (therefore, in the metric $D^{(\rho),W}$ as well) to a function $f(.) \in W(\mathbf{R},\mathcal{U})$ for which $\operatorname{Mod} f \subset \sum_n \operatorname{Mod} f(n;.) \subset \operatorname{Mod} F + \operatorname{Mod} g$. We have $f(n;t) \in F^{(n)}_{j_n}$ and $\operatorname{dist}_{\rho'}(F(t), F^{(n)}_{j_n}) < \gamma_n \epsilon < \frac{1}{6}$ for all $t \in T^{(n)}_{j_n} \bigcap T$. Since $\gamma_n \to 0$ as $n \to +\infty$, it follows from this that $f(t) \in F(t)$ for all $t \in T$ (for a.e. $t \in \mathbf{R}$). With each number $t \in T$ we associate an infinite collection of indices $\{j_1, \dots, j_n, \dots\}$ in such a way that $t \in T^{(n)}_{j_n}$, $n \in \mathbf{N}$. Then (for all $t \in T$)

$$\rho(f(t), g(t)) \leq \sum_{n=1}^{+\infty} \rho(f_{j_1 \dots j_n}, f_{j_1 \dots j_n j_{n+1}}) + \rho(f_{j_1}, g^1_{j_1}) + \rho(g^1_{j_1}, g(t)) <$$

$$< 2 \sum_{n=1}^{+\infty} (\gamma_n + \gamma_{n+1})\epsilon + \frac{\epsilon}{3} + \rho(g^1_{j_1}, F^{(1)}_{j_1}) <$$

$$< \frac{2\epsilon}{3} + |\rho(g^1_{j_1}, F^{(1)}_{j_1}) - \rho(g^1_{j_1}, F(t))| + |\rho(g^1_{j_1}, F(t)) - \rho(g(t), F(t))| + \rho(g(t), F(t)) <$$

$$< \frac{2\epsilon}{3} + \gamma_1\epsilon + \gamma_1\epsilon + \rho(g(t), F(t)) < \epsilon + \rho(g(t), F(t)).$$

□

Remark 3. If conditions of Theorem 5.1 are fulfilled and, moreover, $F \in W_p(\mathbf{R}, \mathrm{cl}_b\mathcal{U}) \subset W(\mathbf{R}, \mathrm{cl}\mathcal{U})$, $p \geq 1$, then it follows from Theorem 2.1 that $f \in W_p(\mathbf{R},\mathcal{U})$. Indeed, for a.e. $t \in \mathbf{R}$ we have

$$\rho(x_0, f(t)) \leq \sup_{x \in F(t)} \rho(x_0, x) = \operatorname{dist}(\{x_0\}, F(t)),$$

furthermore, $\operatorname{dist}(\{x_0\}, F(.)) \in M^\sharp_p(\mathbf{R},\mathbf{R})$. Hence $f(.) \in M^\sharp_p(\mathbf{R},\mathcal{U}) \bigcap W(\mathbf{R},\mathcal{U}) = W_p(\mathbf{R},\mathcal{U})$.

Corollary 5.1. *Let* $(\mathcal{U}, \rho)$ *be a complete separable metric space and let* $F \in W(\mathbf{R}, \mathrm{cl}\mathcal{U})$. *Then there exist functions* $f_j \in W(\mathbf{R},\mathcal{U})$, $j \in \mathbf{N}$, *such that* $\operatorname{Mod} f_j \subset \operatorname{Mod} F$ *and* $F(t) = \overline{\bigcup_j f_j(t)}$ *for a.e.* $t \in \mathbf{R}$ *(if* $F \in W_p(\mathbf{R}, \mathrm{cl}_b\mathcal{U}) \subset W(\mathbf{R}, \mathrm{cl}\mathcal{U})$, $p \geq 1$, *then all functions* f_j *belong to the space* $W_p(\mathbf{R},\mathcal{U})$ *(see Remark 3)).*

Proof. Let us choose points $x_k \in \mathcal{U}$, $k \in \mathbf{N}$, which form a countable dense set of the metric space $\mathcal{U}$. By Theorem 5.1, for all $k, n \in \mathbf{N}$ there are functions $f_{k,n} \in W(\mathbf{R},\mathcal{U})$ such that $\operatorname{Mod} f_{k,n} \subset \operatorname{Mod} F$, $f_{k,n}(t) \in F(t)$ a.e. and $\rho(f_{k,n}(t), x_k) < 2^{-n} + \rho(x_k\,, F(t))$ a.e. It remains to renumber the functions $f_{k,n}(.)$ by a single index $j \in \mathbf{N}$. $\square$

Proof of Theorem 1.3. It can be assumed without loss of generality that $\eta(t) \to 0$ as $t \to +0$. Since $F \in W(\mathbf{R}, \operatorname{cl}\mathcal{U})$ and $g \in W(\mathbf{R},\mathcal{U})$, it follows that $\rho(g(.), F(.)) \in W(\mathbf{R},\mathbf{R})$ and $\operatorname{Mod} \rho(g(.), F(.)) \subset \operatorname{Mod} F + \operatorname{Mod} g$. According to Corollary 3.2, for each $j \in \mathbf{N}$ we choose sets $T_j \in W(\mathbf{R})$ such that $\operatorname{Mod} T_j \subset \operatorname{Mod} \rho(g(.), F(.)) \subset \operatorname{Mod} F + \operatorname{Mod} g$, $\rho(g(t), F(t)) < 2^{-j}$ for all $t \in T_j$, and $\rho(g(t), F(t)) > 2^{-j-1}$ for a.e. $t \in \mathbf{R}\backslash T_j$. Further (after deletion of some subsets of measure zero from the sets T_j, $j \in \mathbf{N}$), we can assume that $T_{j+1} \subset T_j$. Let $T_0 = \mathbf{R}$. We have $T_{j-1}\backslash T_j \in W(\mathbf{R})$ and $\operatorname{Mod} T_{j-1}\backslash T_j \subset \operatorname{Mod} F + \operatorname{Mod} g$, $j \in \mathbf{N}$ (see Lemma 2.5). For each $j \in \mathbf{N}$, according to Theorem 5.1, we choose functions $f_j \in W(\mathbf{R},\mathcal{U})$ for which $\operatorname{Mod} f_j \subset \operatorname{Mod} F + \operatorname{Mod} g$, $f_j(t) \in F(t)$ a.e. and $\rho(f_j(t), g(t)) < \rho(g(t), F(t)) + \eta(2^{-j-1})$ a.e. We define the functions

$$f(.) = \sum_{j=1}^{+\infty} f_j(.)\chi_{T_{j-1}\backslash T_j}(.) + g(.)\chi_{\bigcap_j T_j}(.)\,,$$

$$f(n;.) = \sum_{j=1}^{n} f_j(.)\chi_{T_{j-1}\backslash T_j}(.) + f_{n+1}(.)\chi_{T_n}(.)\,,\ n \in \mathbf{N}\,.$$

By Theorem 1.1, $f(n;.) \in W(\mathbf{R},\mathcal{U})$ and $\operatorname{Mod} f(n;.) \subset \operatorname{Mod} F + \operatorname{Mod} g$. Since $f_j(t) \in F(t)$ a.e. and $g(t) \in F(t)$ for all $t \in \bigcap_j T_j$, it follows that $f(n;t) \in F(t)$ and $f(t) \in F(t)$ a.e. as well. For each $n \in \mathbf{N}$ we have

$$\rho(f(t), f(n;t)) = 0$$

for a.e. $t \in \mathbf{R}\backslash T_{n+1}$,

$$\rho(f(t), f(n;t)) = \rho(f_{m+1}(t), f_{n+1}(t)) \le \rho(f_{m+1}(t), g(t)) + \rho(f_{n+1}(t), g(t)) <$$
$$< 2\,\rho(g(t), F(t)) + \eta(2^{-m-2}) + \eta(2^{-n-2}) < 2^{-n} + 2\,\eta(2^{-n-2})$$

for a.e. $t \in T_m\backslash T_{m+1}$, $m \ge n+1$, and

$$\rho(f(t), f(n;t)) = \rho(g(t), f_{n+1}(t)) < \eta(2^{-n-2})$$

for a.e. $t \in \bigcap_j T_j$. Therefore

$$\operatorname*{ess\,sup}_{t\in\mathbf{R}} \rho(f(t), f(n;t)) \to 0$$

as $n \to +\infty$, hence $f \in W(\mathbf{R},\mathcal{U})$ and $\operatorname{Mod} f \subset \sum_n \operatorname{Mod} f(n;.) \subset \operatorname{Mod} F + \operatorname{Mod} g$. For a.e. $t \in T_{j-1}\backslash T_j$, $j \in \mathbf{N}$, the estimate

$$\rho(f(t),g(t)) = \rho(f_j(t),g(t)) < \rho(g(t),F(t)) + \eta(2^{-j-1}) \leq$$

$$\leq \rho(g(t),F(t)) + \eta(\rho(g(t),F(t)))$$

holds. From this (since $f(t) = g(t) \in F(t)$ for $t \in \bigcap_j T_j$) we obtain that for a.e. $t \in \mathbf{R}$

$$\rho(f(t),g(t)) \leq \rho(g(t),F(t)) + \eta(\rho(g(t),F(t)))\,.$$

If $F \in W_p(\mathbf{R},\mathrm{cl}_b\mathcal{U})$, $p \geq 1$, then also $f \in W_p(\mathbf{R},\mathcal{U})$ (see Remark 3). □

The following Theorems can be proved (using Theorems 1.1 and 1.2, Lemma 2.5 and Corollaries 2.2, 2.3 and 3.2) by analogy with appropriate assertions on Stepanov a.p. functions and multivalued maps [11, 14].

For non-empty set $F \subset \mathcal{U}$ we use the notation $F^{\epsilon} = \{x \in \mathcal{U} : \rho(x,F) < \epsilon\}$, $\epsilon > 0$.

The points $x_j \in \mathcal{U}$, $j = 1,\dots,n$, are said to form ϵ-*net* for (non-empty) set $F \subset \mathcal{U}$, $\epsilon > 0$, if $F \subset \left(\bigcup_j x_j\right)^{\epsilon}$.

Theorem 5.2. *Let $(\mathcal{U},\rho)$ be a complete metric space, let $F \in W(\mathbf{R},\mathrm{cl}_b\mathcal{U})$ and let $\epsilon > 0$, $n \in \mathbf{N}$. Suppose that for a.e. $t \in \mathbf{R}$ there are points $x_j(t) \in F(t)$, $j = 1,\dots,n$, which form ϵ-net for the set $F(t)$. Then for any $\epsilon' > \epsilon$ there exist functions $f_j \in W(\mathbf{R},\mathcal{U})$, $j = 1,\dots,n$, such that* $\operatorname{Mod} f_j \subset \operatorname{Mod} F$, *$f_j(t) \in F(t)$ a.e. and for a.e. $t \in \mathbf{R}$ the points $f_j(t)$, $j = 1,\dots,n$, form ϵ'-net for the set $F(t)$.*

Corollary 5.2. *Let $(\mathcal{U},\rho)$ be a compact metric space. Then a multivalued map $\mathbf{R} \ni t \to F(t) \in \mathrm{cl}\,\mathcal{U} = \mathrm{cl}_b\mathcal{U}$ belongs to the space $W(\mathbf{R},\mathrm{cl}\,\mathcal{U}) = W_1(\mathbf{R},\mathrm{cl}_b\mathcal{U})$ if and only if for each $\epsilon > 0$ there exist a number $n \in \mathbf{N}$ and functions $f_j \in W(\mathbf{R},\mathcal{U}) = W_1(\mathbf{R},\mathcal{U})$, $j = 1,\dots,n$, such that $f_j(t) \in F(t)$ a.e. and points $f_j(t)$, $j = 1,\dots,n$, for a.e. $t \in \mathbf{R}$ form ϵ-net for the set $F(t)$ (furthermore, the functions f_j for the multivalued map $F \in W(\mathbf{R},\mathrm{cl}\,\mathcal{U})$ can be chosen in such a way that* $\operatorname{Mod} f_j \subset \operatorname{Mod} F$*).*

Theorem 5.3. *Let $(\mathcal{U},\rho)$ be a compact metric space. Then a multivalued map $\mathbf{R} \ni t \to F(t) \in \mathrm{cl}\,\mathcal{U}$ belongs to the space $W(\mathbf{R},\mathrm{cl}\,\mathcal{U})$ if and only if there exist functions $f_j \in W(\mathbf{R},\mathcal{U})$, $j \in \mathbf{N}$, such that $F(t) = \overline{\bigcup_j f(t)}$ a.e. and the set $\{f_j(.) : j \in \mathbf{N}\}$ is precompact in the metric space $L_\infty(\mathbf{R},\mathcal{U})$ (furthermore, the functions f_j for the multivalued map $F \in W(\mathbf{R},\mathrm{cl}\,\mathcal{U})$ can be chosen in such a way that* $\operatorname{Mod} f_j \subset \operatorname{Mod} F$*).*

Theorem 5.4. *Let $(\mathcal{U},\rho)$ be a complete metric space, let $F \in W(\mathbf{R},\mathrm{cl}_b\mathcal{U})$, $\epsilon > 0$, $\delta > 0$, $n \in \mathbf{N}$, and let $g_j \in W(\mathbf{R},\mathcal{U})$, $j = 1,\dots,n$. Suppose that for*

a.e. $t \in \mathbf{R}$ *the set of points* $x_j(t) = g_j(t)$, *for which* $g_j(t) \in (F(t))^\delta$, *can be supplemented (if it consists of less than* n *points) to* n *points* $x_j(t) \in (F(t))^\delta$, $j = 1, \dots, n$, *which form* ϵ*-net for the set* $F(t)$ *(coincident points with different indices are considered here as different points). Then for any* $\epsilon' > \epsilon + \delta$ *there exist functions* $f_j \in W(\mathbf{R},\mathcal{U})$, $j = 1, \dots, n$, *such that* $\mathrm{Mod}\, f_j \subset \mathrm{Mod}\, F + \sum_{k=1}^n \mathrm{Mod}\, g_k$, $f_j(t) \in F(t)$ *a.e.,* $f_j(t) = g_j(t)$ *for a.e.* $t \in \{\tau \in \mathbf{R} : g_j(\tau) \in F(\tau)\}$ *and the points* $f_j(t)$, $j = 1, \dots, n$, *for a.e.* $t \in \mathbf{R}$ *form* ϵ'*-net for the set* $F(t)$.

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